Algebraic curves
Solutions sheet 8

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Unless otherwise specified, k is an algebraically closed field.

Exercise 1. Let $r \geq 1$, $P \in \mathbb{A}^r_k$. Call $\mathcal{O} := \mathcal{O}_P(\mathbb{A}^r_k)$ and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$.

1. Compute $\chi(n) = dim_k(\mathcal{O}/\mathfrak{m}^n)$ for $r = 1, 2$.

2. For arbitrary r, show that $\chi(n)$ is a polynomial of degree r in n with leading coefficient $1/r!$.

Solution 1.

1.
   - Let $r = 1$. WLOG we can restrict to the case $P = (0)$. $\mathfrak{m} = (\bar{x})$, $\mathfrak{m}^n = (\bar{x}^n)$. We get $\chi(n) = n$.
   - Let $r = 2$. WLOG take $P = (0, 0)$. $\mathcal{O}_P = k[x, y]_{(x,y)}$. $\mathfrak{m} = (x, y)$. $\mathfrak{m}^n = \oplus_{n \leq i+j} \langle x^i y^j \rangle$, so that

     $$\mathcal{O}/\mathfrak{m}^n = \bigoplus_{i+j<n} \langle x^i y^j \rangle$$

     Hence

     $$\chi(n) = \sum_{k=0}^{n-1} (k+1) = \frac{n(n+1)}{2}$$

2. As before, WLOG take $P = (0, \ldots, 0)$. $\mathcal{O}_P = k[x_1, \ldots, x_r]_{(x_1,\ldots,x_r)}$. $\mathfrak{m} = (x_1, \ldots, x_r)$.

   $$\mathfrak{m}^n = \bigoplus_{n \leq d_1+\cdots+d_r} \langle x_1^{d_1} \cdot \cdots \cdot x_r^{d_r} \rangle$$

   so that

   $$\mathcal{O}/\mathfrak{m}^n = \bigoplus_{d_1+\cdots+d_r<n} \langle x_1^{d_1} \cdot \cdots \cdot x_r^{d_r} \rangle$$

Fixing $0 \leq k \leq n-1$, we count the number of tuples $(d_1, \ldots, d_r)$ such that $d_1 + \cdots + d_r = k$. This is $\binom{k+r-1}{r-1}$, as seen in a previous exercise (with the method of dots and separation).
Then we have the expression

$$\chi(n) = \sum_{k=0}^{n-1} \binom{k+r-1}{r-1}$$

Next we use the identity on a polynomial $P(x) = a_d x^d + \dots$, where $d = degP$, given by :

$$P(x) - P(x-1) = d a_d x^{d-1} + \dots$$

So we can just compute

$$\chi(n) - \chi(n-1) = \binom{n+r-2}{r-1} = \frac{(n+r-2)\dots(n-1)}{(r-1)!}$$

which has degree $r-1$ in n and leading coefficient $\frac{1}{(r-1)!}$. From the previous identity we get the desired answer.

Exercise 2. Find the multiple points and the tangent lines at the multiple points for each of the following curves:

1. $X^4 + Y^4 - X^2Y^2$
2. $X^3 + Y^3 - 3X^2 - 3Y^2 + 3XY + 1$
3. $Y^2 + (X^2 - 5)(4X^4 - 20X^2 + 25)$

Solution 2.

1. $F(X,Y) = X^4 + Y^4 - X^2Y^2$. Let's find multiple points by solving :

$$\frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} = 0$$

We get either $P = (0,0)$, either $X^2 = -Y^2$ in characteristic 3. In all those case $F(X,Y) = 0$ so those are multiple points.

For *char* $k \neq 3$, Using dehomogenisation technique, we see that we have 4 tangent lines at 0, as

$$F(X,Y) = (X - e^{\frac{i\pi}{3}}Y)(X + e^{\frac{i\pi}{3}}Y)(X - e^{\frac{-i\pi}{3}}Y)(X + e^{\frac{-i\pi}{3}}Y)$$

where $e^{\frac{i\pi}{3}}$ is a square root of any primitive third root of unity.

For *char* $k = 3$, this element is not available. And indeed, we have two double tangent lines, as

$$F(X,Y) = (X^2 + Y^2)^2 = (X + iY)^2(X - iY)^2$$

2. $F(X,Y) = X^3 + Y^3 - 3X^2 - 3Y^2 + 3XY + 1$. If (X,Y) is a multiple point, then :

$$\frac{\partial F}{\partial X} = 3X^2 - 6X + 3Y = 0$$

$$\frac{\partial F}{\partial Y} = 3Y^2 - 6Y + 3X = 0$$

In *char* $k \neq 3$, we compute : $X^2 = 2X - Y$, $Y^2 = 2Y - X$. So $X^3 = 2X^2 - XY = 4X - XY - 2Y$, $Y^3 = 4Y - XY - 2Y$. Plugging in $F(X,Y) = 0$, we get

$$F(X,Y) = 4X - XY - 2Y + 4Y - XY - 2 - 3(2X - Y) - 3(2Y - X) + 3XY + 1 = 0$$

Hence $(X-1)(Y-1) = 0$, so either $X = 1$, either $Y = 1$. From that we see that there is a single multiple point $P = (1,1)$.

Then we compute

$$F(X+1, Y+1) = 3XY + X^3 + Y^3$$

So that $m_P = 2$ and there are two simple tangent line X and Y (it is a node).

In characteristic 3, all points are multiple with a triple tangent line $Y = -X$.

3. $F(X,Y) = Y^2 + (X^2 - 5)(4X^4 - 20X^2 + 25)$. Let (X,Y) be a multiple point. In *char* $k \neq 2$, we get $\frac{\partial F}{\partial Y} = 2Y = 0$ so $Y = 0$. Then,

$$F(X,Y) = (X^2 - 5)(4X^4 - 20X^2 + 25) = (X^2 - 5)(2X^2 - 5)^2$$

In *char* $k \neq 5$, there are two multiple roots, given by $X = \pm\sqrt{\frac{5}{2}}$ and $Y = 0$, with distinct tangent lines.

In *char* $k = 5$, get $F(X,Y) = Y^2 - X^6$. Multiple point at $(0,0)$ with one double tangent line.

In *char* $k = 2$, get $F(X,Y) = Y^2 + X^2 + 1 = (Y + X + 1)^2$. All points of the curve are multiple points, with double tangent line.

Exercise 3. Let $T : \mathbb{A}^2_k \to \mathbb{A}^2_k$ be a polynomial map, $Q \in \mathbb{A}^2_k$ and $P = T(Q)$. If T is written component-wise as (T_1, T_2), the Jacobian matrix of T at Q is defined as $J_Q(T) = (\partial T_i / \partial X_j(Q))_{1 \leq i,j \leq 2}$.

1. Show that $m_Q(F^T) \geq m_P(F)$.
2. Show that if $J_Q(T)$ is invertible, then $m_Q(F^T) = m_P(F)$.
3. Show that the converse of the previous statement is false.

Solution 3.

1. Just from definition,

$$\begin{aligned} F^T(x_Q + X, y_Q + Y) &= F(x_P + \textstyle\sum_{(i,j)\neq(0,0)} a_{i,j} X^i Y^j, y_P + \sum_{(i,j)\neq(0,0)} b_{i,j} X^i Y^j) \\ &= F^{T_P}(\textstyle\sum_{(i,j)\neq(0,0)} a_{i,j} X^i Y^j, \sum_{(i,j)\neq(0,0)} b_{i,j} X^i Y^j) \\ &= F_m \circ G + \dots \end{aligned}$$

where $G = \sum_{i,j\neq(0,0)} \frac{1}{(i+j)!} \frac{\partial^{i+j} T}{\partial x^i \partial y^j}(Q) X^i Y^j$ is polynomial without constant terms. Hence $m_Q(F^T) \geq m_P(F)$.

2. The first term in G is $J_Q(T)\binom{X}{Y}$. If $J_Q(T)$ is invertible, both lines are non zero, hence $m_Q(F^T) = m_P(F)$.
3. A counterexample is given by $F = X^2 - X^3$, $P = Q = (0,0)$, $T(X,Y) = (X^2, Y)$. Then $m_Q(F^T) = m_P(F) = 2$ but $J_Q(T) = \begin{pmatrix} x_Q & 0 \\ 0 & 1 \end{pmatrix}$ is not invertible.

Exercise 4. Let $n \geq 2$ and $F \in k[X_1, \dots, X_n]$. Consider $V(F) \subseteq \mathbb{A}^n_k$ the associated hypersurface and $P \in V(F)$.

1. Define the multiplicity $m_P(F)$ of F at P.
2. If $m_P(F) = 1$, define the tangent hyperplane of F at P.
3. Can you define tangent hyperplanes for $F = X^2 + Y^2 - Z^2$ at $P = (0,0,0)$?

4. Assume that F is irreducible. Show that, for $n = 2$ (curves), $V(F)$ has finitely many multiple points. Is this true for $n > 2$?

Solution 4.

1. Let $P = (0, \ldots, 0)$. We can define $m_P(F)$ as in the plane curve case. It is the smallest degree m of a summand in the decomposition of F into linear forms. Then for a general P, $m_P(F) = m_0(F^T)$.

2. If $m_P(F) = 1$, then $F^T = F_1 + \ldots$. The tangent hyperplane of F at P is $V(F_1)$.

3. In this example, $m_P(F) = 2$, and for any deshomogenisation, we get a non factorizable polynomial, e.g. $X^2 + Y^2 - 1$. Geometrically it corresponds to the fact that this equation defines an infinite cone whose summit is the origin.

4. Let F be irreducible. Suppose F have infinitely many multiple points. Then $deg\ F \geq 2$. Then $\frac{\partial F}{\partial X}$ and $\frac{\partial F}{\partial Y}$ vanishes at infinitely many point in $V(F)$, hence their zero locus is one dimensional, hence they vanish on $V(F)$ because F is irreducible. But then F divide them, so that they are 0, which contradicts the degree 2.

   For $n > 2$, we have a counterexample given by taking the product of any irreducible nodal curve with $\mathbb{A}^1$.

Exercise 5. Let $R = k[\epsilon]/(\epsilon^2)$ and $\varphi : R \to k$ the k-algebra homomorphism sending ϵ to 0 (R is often called the ring of *dual numbers*). Let $F \in k[X, Y]$ irreducible, $P \in V(F)$, $\mathfrak{m}_P \subseteq \Gamma(F)$ the corresponding maximal ideal and $\theta_P : \Gamma(F) \to \Gamma(F)/\mathfrak{m}_P \simeq k$ the associated k-algebra homomorphism.

1. Suppose that P is a simple point. Show that there is a bijection between the tangent line to F at P and $\{\theta \in Hom_{k-alg}(\Gamma(F), R) \mid \varphi \circ \theta = \theta_P\}$.

2. What happens for multiple points (for instance, $F = Y^2 - X^3$, $P = (0, 0)$)?

Solution 5.

1. Let θ be such that $\varphi \circ \theta = \theta_P$. It means that θ corresponds to a pair $(a, b) \in k^2$ such that

   $$F(x_P + \epsilon a, y_P + \epsilon b) = 0$$

   Then, we compute that $F(x_P + \epsilon a, y_P + \epsilon b) = F(x_P, y_P) + J_P(F)\binom{a}{b}$.

   $F(x_P, y_P) = 0$. We get $J_P(F)\binom{a}{b} = 0$ which is $(a, b) \in T_P(V(F))$.

2. At the cusp, for all $a, b \in k^2$, $F(a\epsilon, b\epsilon) = 0$ because $\epsilon^2 = 0$. It corresponds to the fact that the whole plane is the Zariski tangent space at the cusp.